Physics 222 Hints #1: Remarks on the Biot-Savart Law

The PHY 222 textbook by Serway gives **two** versions of the **Biot-Savart** law:

$$B = \frac{\mu_0 I}{2\pi a} \tag{19.15}$$

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\ d\vec{s}\times\hat{r}}{r^2} \quad \text{which implies} \quad dB = \frac{\mu_0}{4\pi}\frac{I\ ds\ \sin\theta}{r^2} \tag{19.13}$$

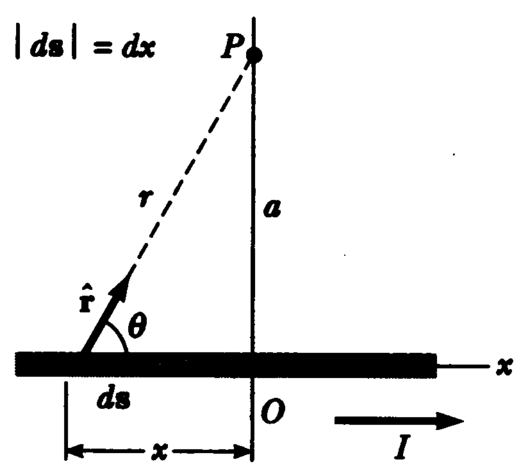
Both equations give the **strength** (magnitude) B or dB of the magnetic field due to a wire (current-carrying conductor). The **direction** of the magnetic field can be derived from the right-hand rule.

I want to answer the following questions here:

- **What is the difference between these equations ?**
  This questions is easy to answer: Eq. (19.15) gives the magnitude of the magnetic field due to an infinitely long straight wire at a distance a from the wire, if a current I is flowing through the wire. Eq. (19.13), on the other hand, is the field due to a small section with length ds of a wire. Refer to Fig. 19.13 (a) for the meaning of the parameters r and θ.

- **When do I need to use one and when the other ?**
  This can also be explained easily: You should use Eq. (19.15) whenever you deal with an infinitely long straight wire, since this equation is easier to use. If the wire is not infinitely long, however, or if it is not straight, then you **cannot** use Eq. (19.15). In this case, you need to calculate the field by dividing the wire into infinitesimal sections of length ds, calculate the contribution dB to the magnetic field by each segment from Eq. (19.13), and then sum (or integrate) over all these contributions.

- **What is the relationship between these equations ? If one is a special case of the other, should it not be possible to derive one from the other ?**
  Serway explains this relationship in Example 19.4. However, I don't seem to remember the definition of the csc function. Besides that, I prefer to integrate over the length x of the wire rather than the angle θ. This seems more natural to me.
  The expression for the field of an infinitely long wire, Eq. (19.15), can be derived from the Biot-Savart law (19.13) with one moderately difficult integration. This is explained in the following. Please refer to Fig. 19.13 (a) of Serway (to the left).



(a)

The magnitude of the field due to a current element (piece of wire) dl in Fig. 30.5 is given by Biot-Savart's law

$$dB = \frac{\mu_0}{4\pi}\frac{I\ ds\ \sin\theta}{r^2}. \tag{1}$$

In this equation, $\vec{r}$ is the vector pointing from the current element ds to the point where we calculate the magnetic field. r is the magnitude (length) of this vector. θ is the angle between the wire and the vector $\vec{r}$. a is the shortest distance from the point where we calculate the magnetic field to the wire. x is the projection of $\vec{r}$ onto the wire. In other words, a and x are the legs of a triangle and r is its hypotenuse. We use the Pythagorean theorem and basic trigonometry to show that $r^2 = x^2 + a^2$ and $\sin\theta = a/r$. Then, the Biot-Savart law takes the form

$$dB = \frac{\mu_0}{4\pi}\frac{I\ ds\ \sin\theta}{x^2+a^2} = \frac{\mu_0}{4\pi}\frac{I\ ds\ a}{(x^2+a^2)^{3/2}} \tag{2}$$

Now we have the contribution of one current element ds to the magnetic field. We need to add (integrate) all these contributions together. Note that $ds = dx$.

$$B(\vec{r}) = \int dB = \int_{-\infty}^{\infty} \frac{\mu_0}{4\pi}\frac{I\ dx\ a}{(x^2+a^2)^{3/2}} = \frac{\mu_0 I a}{4\pi}\int_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)^{3/2}}. \tag{3}$$

I certainly do **not** claim that I know how to solve such integrals without looking them up in a table. For such purposes, the Appendix B (Mathematics Review) of Serway comes in handy. On page A.26, Table B.5, we find among other things:

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}} \tag{4}$$

With this formula, we can solve the integral and find the strength of the magnetic field for an infinitely long straight wire:

$$B(\vec{r}) = \frac{\mu_0 I a}{4\pi}\int_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)^{3/2}} = \frac{\mu_0 I a}{4\pi}\left.\frac{x}{a^2\sqrt{x^2+a^2}}\right|_{-\infty}^{+\infty} = \frac{\mu_0 I}{4\pi a}[1-(-1)] = \frac{\mu_0 I}{2\pi a}. \tag{5}$$

We have now derived the magnetic field of an infinitely long straight wire from the differential (current element) form of the Biot-Savart law.

Dr. Stefan Zollner, A205 Physics, 4-7327. August 27, 1995
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